Solutions for Exam Physics Laboratory 1: Data and error analysis
3 February 2017

Exercise 1

(5 points total)
(For parts a) - d), the answer should be exactly correct to get 1 point, if something is incorrect, then give 0 points)

a) $\Delta v = 0.3$ km/s so $v = 7.4 \pm 0.3$ km/s
(1 point)

b) $\Delta\lambda = 0.04$ μm so $\lambda = 0.38 \pm 0.04$ μm $= (3.8 \pm 0.4) \times 10^2$ nm
(1 point)

c) $\Delta T = 0.3$ mK so $T = 5764150.0 \pm 0.3$ mK $= 5764.1500 \pm 0.0003$ K (the relatively small error warrants more significant digits than originally given)
(1 point)

d) $\Delta p = 0.02$ MPa so $p = 2.89 \pm 0.02$ MPa
(1 point)

e) Probability $P = f(k) = \frac{\lambda^k}{k!}e^{-\lambda}$ with $\lambda = 4.0$ and $k = 2$ so $P = \frac{4.0^2}{2!}e^{-4.0} = 0.147$ so $P = 15\%$ so answer **B**. Any calculation or argument may be ignored, but if a student starts OK but makes a small mistake later on, then 0.5 point may be given.
(1 point)

Exercise 2

(7 points total)

a)

$$\begin{aligned} Z &= \sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2} = \sqrt{5600^2 + \left(\frac{1}{2\pi \cdot 50 \cdot 470 \cdot 10^{-9}}\right)^2} \\ &= \sqrt{5600^2 + 6772.55^2} = 8787.9\ \Omega \end{aligned}$$

[some rounding is allowed here but not necessary, because this will be required in part c); subtract 0.5 point if unit Ω is forgotten]
(1 point)

b) Method of substitution: substitute $X = \left(\frac{1}{2\pi f C}\right)^2 \Rightarrow X = 6772.55^2 = 4.5867 \cdot 10^7$ Ω^2 and $(\Delta X/X) = |-2(\Delta C/C)| \Rightarrow \Delta X = 2 \cdot \frac{33}{470} \cdot X = 6.441 \cdot 10^6\ \Omega^2$.
Substitute $V = R^2 + X$, then $V = Z^2 = 7.7227 \cdot 10^7\ \Omega^2$ and
$\Delta V = \sqrt{(2R\Delta R)^2 + (\Delta X)^2} = \sqrt{[2 \cdot 5600 \cdot (5\% \cdot 5600)]^2 + (6.441 \cdot 10^6)^2}$
$= 7.1639 \cdot 10^6\ \Omega^2$. Now $Z = \sqrt{V}$,
so $(\Delta Z/Z) = \frac{1}{2}(\Delta V/V) = \frac{1}{2} \cdot (7.1639 \cdot 10^6 / 7.7227 \cdot 10^7) = 0.0464 \approx 4.6\%$ and
$\Delta Z = 0.0464 \cdot Z = 407.6\ \Omega \approx 0.5$ kΩ.

[some rounding is allowed here but not necessary, because this will be required in part c)]

Method of partial derivatives:

$$\frac{\partial Z}{\partial R} = \frac{2R}{2\sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2}} = \frac{R}{\sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2}} = \frac{R}{Z}$$

and

$$\frac{\partial Z}{\partial C} = \frac{-2\left(\frac{1}{2\pi f}\right)^2 C^{-3}}{2\sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2}} = \frac{-1}{(2\pi f)^2 C^3 \sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2}} = \frac{-1}{(2\pi f)^2 C^3 Z}$$

so

$$(\Delta Z)^2 = \left(\frac{\partial Z}{\partial R}\Delta R\right)^2 + \left(\frac{\partial Z}{\partial C}\Delta C\right)^2 = \left(\frac{R}{Z}\Delta R\right)^2 + \left(\frac{\Delta C}{(2\pi f)^2 C^3 Z}\right)^2$$

so $\Delta Z = \sqrt{31836.14 + 134298.01} = \sqrt{166134.15} = 407.60\ \Omega$, from which follows $(\Delta Z/Z) = 0.0464 \approx 4.6\%$.

[3 points for correct method/formulae (either substitution or partial derivatives);
1 point for correct value of ΔZ and
1 point for correct value of $\Delta Z/Z$;
subtract 0.5 point if unit Ω is forgotten in ΔZ]
(5 points total)

c) $Z = 8787.9\ \Omega$ and $\Delta Z = 407.6\ \Omega$ [from a) and b)] so
$Z = 8.8 \pm 0.5$ kΩ is the correct notation. *[subtract 0.5 point if unit Ω or* kΩ *is forgotten]*
(1 point)

Exercise 3

(4 points total)

a) $m_1 = 37.3 \pm 0.2$ g, $m_2 = 37.4 \pm 0.1$ g, $m_3 = 37.2 \pm 0.3$ g, so
$w_1 = s_1^{-2} = 25$, $w_2 = s_2^{-2} = 100$, $w_3 = s_3^{-2} = 11.1111$.
$m_{avg} = \frac{\sum w_i m_i}{\sum w_i} = 37.3653 \approx 37.37$ g. *(1 point for correct weights + 1 point for correct weighted average; if the answer contains more than 2 decimal places, subtract 0.5 point; subtract 0.5 point if unit forgotten)*
(2 points total)

b) $\frac{1}{s_{m_{avg}}}^2 = \sum w_i = 136.1111 \Rightarrow s_{m_{avg}} = 136.1111^{-1/2} = 0.0857 \approx 0.09$ g (round error upward). In the correct notation: $m_{avg} \pm \Delta m = 37.37 \pm 0.09$ g (but this notation is not required here)
(2 points total; subtract 0.5 point if unit forgotten)

Exercise 4

(4 points total)

a) $z_1 = (\rho_1 - \rho_{av})/\sigma = (3.503 - 3.515)/0.015 = -0.012/0.015 = -0.8$ and $z_2 = (\rho_2 - \rho_{av})/\sigma = (3.536 - 3.515)/0.015 = 0.021/0.015 = 1.4$ so probability $P = \int_{z_1}^{z_2} N(y)dy = \int_{-0.8}^{1.4} N(y)dy = \int_0^{0.8} N(y)dy + \int_0^{1.4} N(y)dy \approx 0.2881 + 0.4192 = 0.7073$ so $P \approx 71\%$. *(1 point for correct method, 1 point for correct numbers)*
(2 points total)

b) $z_3 = (\rho_3 - \rho_{av})/\sigma = (3.497 - 3.515)/0.015 = -0.018/0.015 = -1.2$ so probability $P = \int_{-\infty}^{z_3} N(y)dy = \int_{-\infty}^{-1.2} N(y)dy = \int_{1.2}^{\infty} N(y)dy = \frac{1}{2} - \int_0^{1.2} N(y)dy \approx \frac{1}{2} - 0.3849 = 0.1151$ so $P \approx 12\%$. *(1 point for correct method, 1 point for correct numbers)*
(2 points total)

Exercise 5

(9 points total)

a) $\overline{T} = (1/5) \times (4.2 + 4.1 + 4.4 + 4.5 + 4.0) = 4.24 \approx 4.2$ K. *(4.24 is one digit too many, given the number of decimals in the input data: subtract $\frac{1}{2}$ point)*
(2 points total)

b) $N = 5; s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{5}(T_i - \overline{T})^2} = \sqrt{\frac{1}{4}((-0.04)^2 + (-0.14)^2 + 0.16^2 + 0.26^2 + (-0.24)^2)} = \sqrt{\frac{1}{4}0.1720} = \sqrt{0.043} = 0.2074 \approx 0.3$ K *(round error upward, subtract 1 point if this is not done; subtract 0.5 point if units are not given).*
(2 points)

c) $s_m = s/\sqrt{N} = 0.2074/\sqrt{5} = 0.0927 \approx 0.1$ K. (round error upward). If $s_m = s/\sqrt{N} = 0.3/\sqrt{5} = 0.134 \approx 0.2$ K is given, this is also counted as correct. *(round error upward, subtract 1 point if this is not done; subtract 0.5 point if units are not given).*
(2 points)

d) 175 measurements extra, so in total 175 + 5 = 180 measurements. The error s_m is smaller by a factor of $\sqrt{180/5} = \sqrt{36} = 6$.
(2 points)

e) No, because very many extra measurements are needed, which takes a lot of time and effort. It is better to use a more accurate method to begin with.
(1 point)

Exercise 6

(9 points total)

$$a = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{N\sum x_i^2 - \left(\sum x_i\right)^2},$$

$$(\Delta a)^2 = \left(\frac{1}{\sum x_i^2 - N\overline{x}^2}\right)\frac{\sum r_i^2}{N-2},$$

x	$y \pm \Delta y$	r
-2	5 ± 2	0.6780
-1	2 ± 3	-1.0678
1	1 ± 1	0.4407
3	-2 ± 2	-0.0508

Table 1: Observations for exercise 6, now with residuals.

$$(\Delta b)^2 = \left(\frac{1}{N} + \frac{\overline{x}^2}{\sum x_i^2 - N\overline{x}^2}\right)\frac{\sum r_i^2}{N-2}.$$

a) $N = 4, \sum x_i = 1, \sum y_i = 6, \sum x_i^2 = 15, \sum x_i y_i = -17$
$\Rightarrow a = (4 \cdot -17 - 1 \cdot 6)/(4 \cdot 15 - \{1\}^2) = -1.2542 \approx -1.3$
and $\overline{x} = 0.25, \overline{y} = 1.5 \Rightarrow b = \overline{y} - a\overline{x} = 1.8136 \approx 1.8$. *(if values for a and b are not rounded to 1 or 2 decimal places, subtract 0.5 point just once)*
(2 points total)

b) $\sum r_i^2 = 1.7966 \Rightarrow (\Delta a)^2 = (15 - 4 \cdot 0.25^2)^{-1}(1.7966/\{4-2\}) = 0.0609$
$\Rightarrow \Delta a = 0.2468 \approx 0.3$ *(rounded upward)*
and $(\Delta b)^2 = (1/4 + 0.25^2/\{15 - 4 \cdot 0.25^2\})(1.7966/\{4-2\}) = 0.2284$
$\Rightarrow \Delta b = 0.4779 \approx 0.5$ *(rounded upward) (if values for Δa and Δb are not rounded up to 1 decimal place, subtract 0.5 point just once)*
(2 points total)

If substitute answers $a = -1.5$ and $b = 2$ are used, then $r = (0, -1.5, 0.5, 0.5)$ and $\sum r_i^2 = 2.75$ so $\Rightarrow (\Delta a)^2 = (15 - 4 \cdot 0.25^2)^{-1}(2.75/\{4-2\}) = 0.0932$
$\Rightarrow \Delta a = 0.3053 \approx 0.4$ *(rounded upward)*
and $(\Delta b)^2 = (1/4 + 0.25^2/\{15 - 4 \cdot 0.25^2\})(2.75/\{4-2\}) = 0.3496$
$\Rightarrow \Delta b = 0.5912 \approx 0.6$ *(rounded upward)*

c) $\chi^2_{obs} = \sum\{r_i/(\Delta y_i)\}^2 = (0.6780/2)^2 + (-1.0678/3)^2 + (0.4407/1)^2 + (-0.0508/2)^2 = 0.4365$. *(subtract 1 point for forgetting squares in formula; no points at all if summation is forgotten; subtract 1 point if Δy = constant is used; subtract 1.5 point if mistake in calculation; if only the correct formula is written without any further calculation, give 0.5 point for knowing the formula)*
(3 points total)

If substitute answers $a = -1.5$ and $b = 2$ are used, then $r = (0, -1.5, 0.5, 0.5)$ so $\chi^2_{obs} = (0/2)^2 + (-1.5/3)^2 + (0.5/1)^2 + (0.5/2)^2 = 0.5625$.

d) 2 parameters a and b are fitted, so $\nu = N - 2 = 2$. *(1 point)*
1% level $\Rightarrow \chi^2_{table} = 0.020$, 99% level $\Rightarrow \chi^2_{table} = 9.210$. χ^2_{obs} is in between those two limits, so the linear fit is acceptable. *(1 point)*
(2 points total)

If substitute answers $a = -1.5$ and $b = 2$ are used, then $\chi^2_{obs} = 0.5625$, which is still in between the two limits, so a linear fit is still acceptable in this case.